

# Invariant Gaussian Fields on Homogeneous Spaces : Explicit Constructions and Geometric Measure of the Zero-Set

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## Abstract

This paper is concerned with the properties of Gaussian random fields defined on a riemannian homogeneous space, under the assumption that the probability distribution be invariant under the isometry group of the space. We first indicate, building on early results on Yaglom, how the available information on group-representation-theory-related special functions makes it possible to give completely explicit descriptions of these fields in many cases of interest. We then turn to the expected size of the zero-set: extending two-dimensional results from Optics and Neuroscience, we show that every invariant field comes with a natural unit of volume (defined in terms of the geometrical redundancies in the field) with respect to which the average size of the zero-set depends only on the dimension of the source and target spaces, and not on the precise symmetry exhibited by the field. Both the volume unit and the associated density of zeroes can in principle be evaluated from a single sample of the field, and our result provides a numerical signature for the fact that a given individual map be a sample from an invariant Gaussian field.

## 1 Introduction

Interest for Gaussian random fields with symmetry properties has risen recently. While it is not surprising that these fields should have many applications (Kolmogorov insisted as early as 1944 that they should be relevant to mathematical discussions of turbulence), a short list of recent domains in which they appeared will help me describe the motivation for this paper.

- Optics and the Earth sciences. Suppose a wave is emitted at some point, but thereafter undergoes multiple diffractions within a disordered material (like a tainted glass, or the inside of the Earth). If the material is disordered enough, beyond it one will observe a superposition of waves propagating in somewhat random directions, with somewhat random amplitudes and phases. In Sismology and in Optics, there are theoretical and practical benefits in treating the output as a single realization of a complex-valued gaussian field, which inherits symmetry properties from those of the material which diffracted the waves.

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See [10] for Optics, [36] for Sismology.

- Astrophysics. The Cosmic Microwave Background (CMB) radiation is understood to be an observable relic of the “Big Bang”. There are fluctuations within it<sup>1</sup>: the frequency of the radiation changes very slightly around the mean value as one looks to different parts of the sky. The fact that the variations are small is essential for cosmology of course, but their precise structure is quite as important: they are supposed to be a relic of the slight homogeneities which made it possible for the galaxies and stars to take shape. Because it is customary to assume that the universe is, and has always been, isotropic on a large scale, a much-discussed model for the CMB treats it as a single realization of an isotropic gaussian field on the sphere<sup>2</sup>. This has of course prompted mathematical developments [21], as well as motivation for a recent monograph by A. Malyarenko [20] on random fields with symmetry properties.

- Texture modelling and synthesis. Many homogeneous-looking regions in natural images, usually called *textures*, are difficult to distinguish with the naked eye from realizations of an appropriate stationary Gaussian random field. Thus, stationary Gaussian fields are a simple and natural tool for image synthesis; an appreciable advantage of this is that thanks to the ergodicity properties of Gaussian fields ([2], chapter 6), the probability distribution of a stationary Gaussian random field can be roughly recovered from a single realization: measuring correlations in a single sample yields a good approximation for the covariance function of the field, and one can draw new examples of the given texture from it: this is widely used in practice. See [30, 26, 14]. Textures obviously have a meaning on homogeneous spaces as well as Euclidean space<sup>3</sup>, and the mathematical generalization of widely-used image-processing tools to curved spaces should feature homogeneous fields<sup>4</sup>.

- Neuroscience (more detailed discussions can be found in [3, 4]). In the primary visual cortex of mammals, neurons record several local features of the visual input, and the electrical activity of a given neuron famously depends on the presence, in its favourite region of space, of oriented stimuli (“edges”): there are neurons which activate strongly in the presence of vertical edges in the image, others which react to oblique edges, and so on. The map which, to a point of the cortical surface, assigns the “orientation preference” of the neuron situated there (the stimulus direction which maximizes the electrical activity of the neuron) has been observed to be continuous in almost all (though not all) mammals, and to have strikingly constant geometrical properties across species and individuals. Prevailing models for the early stage in the development of these cortical maps treat the arrangement in a given individual as a single realization of a Gaussian random field on the cortical surface, with the orientation map obtained after taking the argument; a key to the success of the models is the assumption, meant to reflect the initial homogeneity of the biological tissue, that when the cortical surface is identified with a Euclidean plane, the underlying

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<sup>1</sup>These were detected in 1992; see G. Smoot and J. Mather’s Nobel lectures [27, 22].

<sup>2</sup>In fact, the expectation function of the field should not be a constant, because the CMB has rather large-scale fluctuations, famously attributed to a Doppler effect due to the metric expansion of space, in addition to the above-mentioned variations. So it is the centered version (with the large-scale fluctuations subtracted off) that should be isotropic. Instead of being Gaussian, the field could also be a function of an underlying Gaussian field: this seems to be a prevailing hypothesis; see [21].

<sup>3</sup>The example images in [14] can obviously be imagined on the surface of a sphere or in a hyperbolic plane.

<sup>4</sup>This does not mean that it will be faster to work with on a computer than a version less naturally suited to the curvature of the manifold!

Gaussian field is homogeneous and isotropic.

In at least two of these fields, Optics and Neuroscience, the *zeroes* of stationary gaussian fields have attracted detailed attention. In a heated debate on the evolution of the early visual system in mammals (see [23]), the mean number of zeroes in a region with a given area has been taken up as a criterion to decide between two classes of biological explanations for the geometry of cortical maps. Experiments show the mean value to be remarkably close to  $\pi$  with respect to an appropriate unit of area (re-defined for Gaussian fields in section 5 below). This strikingly coincides with the exact mean value obtained for stationary isotropic Gaussian fields by Wolf and Geisel in related work, and independently by Berry and Dennis in optics-related work (there the zeroes are points where the light goes off, or the sound waves cancel each other : Berry and Dennis call them "lines of darkness, or threads of silence"). Along with the key role symmetry arguments play in the discussion of the visual cortex, the remarkable coincidence is one of my motivations for generalizing to arbitrary homogeneous spaces the Euclidean-and-planar results which appeared in Optics and Neuroscience.

These recent developments take up an old theme: understanding the properties of the level sets of (the paths of) a random field is a classical subject in the theory of stochastic processes.

This paper is a mathematical follow-up on [3] ; some of the results below have been announced (with incorrect statements !) in the appendix to that article. It has two relatively independent aims:

- Describe invariant Gaussian fields on homogeneous spaces as explicitly as possible,
- Study the mean number of zeroes, or the average size of the zero-set, of an invariant field in a given region of a homogeneous space.

The first problem has been solved in the abstract by Yaglom in 1961 [35] using the observation (to be recalled in section 2.1) that the possible correlation functions of homogeneous complex-valued fields form a class which has been much studied in the representation theory of Lie groups. Section 2.2 is a summary (with independent proofs) of the consequences of his results that I will use. Since Yaglom's time, representation theory has grown to incorporate several more concrete constructions, and in section 3 below, I show that explicit descriptions (that can be worked with on a computer) are possible on many spaces of interest, including symmetric spaces. Section 3.1 also includes simple facts which show that on a given manifold, not all transitive Lie groups can give rise to invariant random fields with continuous trajectories.

Turning to the second problem, what I show below is that when expressed in a unit of volume appropriate to the field (defined in section 4 for real-valued fields and at the beginning of section 5 for others), the average size of the zero-set does not depend on the group acting, but only on the dimension of the homogeneous space on which the field is defined and that of the space in which it takes its values. When looking at a single realization of a random field, observing the average size for the zero-set expressed by Theorem 2 below can be viewed a signature that the field has a symmetry, regardless of the fine structure of the symmetry involved.

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## 2 Invariant real-valued gaussian fields on homogeneous spaces

### 2.1 Gaussian fields and their correlation functions

Suppose  $X$  is a smooth manifold and  $V$  a finite-dimensional Euclidean space. A *Gaussian field on  $X$  with values in  $V$*  is a random field  $\Phi$  on  $X$  such that for each  $n$  in  $\mathbb{N}$  and every  $n$ -tuple  $(x_1, \dots, x_n)$  in  $X^n$ , the random vector  $(\Phi(x_1), \dots, \Phi(x_n))$  in  $V^n$  is a Gaussian vector. A Gaussian field  $\Phi$  is *centered* when the map  $x \mapsto \mathbb{E}[\Phi(x)]$  is identically zero, and it is *continuous*, resp. *smooth*, when  $x \mapsto \Phi(x)$  is almost surely continuous, resp. smooth.

In this paper, our space  $X$  will be a smooth manifold equipped with a smooth and transitive action  $(g, x) \mapsto g \cdot x$  of a Lie group  $G$ . Choose  $x_0$  in  $X$  and write  $K$  for the stabilizer of  $x_0$  in  $G$ . A Gaussian field on  $X$  with values in  $V$  is *invariant* when the probability distribution of  $\Phi$  and that of the Gaussian field  $\Phi \circ (x \mapsto g \cdot x)$  are the same for every  $g$  in  $G$ .

The case in which  $V$  equals  $\mathbb{R}$  is of course important. If  $\Phi$  is a real-valued Gaussian field on  $X$ , its *covariance function* is the (deterministic) map  $(x, y) \mapsto \mathbb{E}[\Phi(x)\Phi(y)]$  from  $X \times X$  to  $\mathbb{R}$ . A real-valued Gaussian field is *standard* if it is centered and if  $\Phi(x)$  has unit variance at each  $x \in X$ .

When describing scalar-valued Gaussian fields with symmetry properties, we shall see in the next subsection that the relationship with representation theory makes it useful that the covariance function, and thus the field as well, be allowed to be complex-valued rather than real-valued. A precise word about the kind of complex-valued Gaussian fields we need is perhaps in order here.

A *circularly symmetric Gaussian variable* is a complex-valued random variable whose real and imaginary parts are independent, identically distributed real Gaussian variables. A *circularly symmetric complex Gaussian field on  $X$*  is a Gaussian random field  $Z$  on  $X$  with values in the vector space  $\mathbb{C}$ , with the additional requirement that  $(x, y) \mapsto \mathbb{E}[Z(x)Z(y)]$  be identically zero. Note that while this imposes that  $Z(x)$  be circularly symmetric for all  $x$ , this does not necessitate that  $\Re(Z)(x)$  and  $\Im(Z)(y)$  be uncorrelated if  $x$  is not equal to  $y$ .

The *correlation, or covariance, function* of a circularly symmetric complex Gaussian field on  $X$  is the (deterministic) map  $(x, y) \mapsto \mathbb{E}[Z(x)Z(y)^*]$  from  $X \times X$  to  $\mathbb{C}$ , where the star denotes complex conjugation. A *standard complex Gaussian field on  $X$*  is a circularly symmetric complex Gaussian field on  $X$  such that  $\mathbb{E}[Z(x)Z(x)^*] = 1$  for all  $x$ .

Note that the real part of the covariance function of a circularly symmetric complex-valued Gaussian field is twice the covariance function of the real-valued Gaussian field obtained by considering its real part. A circularly symmetric complex Gaussian field on  $X$  has a real-valued correlation function if and only if its real and imaginary parts are independent *as processes*.

We are now ready for the classical theorem which describes the correlation functions of standard complex Gaussian fields, those of real-valued Gaussian fields being a particular case as we saw (see however [1] for a separate description of the real case).

**Proposition 2.1** (see for instance [18], section 2.3). *Suppose  $C$  is a deterministic map from  $X \times X$  to  $\mathbb{C}$ . Then it is the covariance function of a continuous (resp. smooth), invariant, standard complex-valued Gaussian field if and only if it has the following properties.*

- (a) *The map  $C$  is continuous (resp. smooth);*
- (b) *for each  $x, y$  in  $X$  and every  $g$  in  $G$ ,  $C(gx, gy) = C(x, y)$ ;*
- (c) *for every  $x$  in  $X$ ,  $C(x, x) = 1$ ;*
- (d) *(positive-definiteness) for each  $n$  in  $\mathbb{N}$  and every  $n$ -tuple  $(x_1, \dots, x_n)$  in  $X^n$ , the hermitian matrix  $(C(x_i, x_j))_{1 \leq i, j \leq n}$  is positive-definite.*

*If  $\Phi_1$  and  $\Phi_2$  are continuous (resp. smooth), invariant, standard complex-valued Gaussian fields with covariance function  $C$ , then they have the same probability distribution.*

A consequence is that there is a left-and-right  $K$ -invariant continuous (resp. smooth) function  $\Gamma$  on  $G$ , taking the value one at  $1_G$ , such that  $C(gx, x) = \Gamma(g)$  for every  $g$  in  $G$  and every  $x$  in  $X$ . Proposition 2.1 thus says that taking covariance functions yields a natural bijection between

- probability distributions of continuous (resp. smooth), invariant, standard complex-valued Gaussian fields on  $X = G/K$ ,
- and
- positive-definite, continuous (resp. smooth),  $K$ -bi-invariant functions on  $G$ , taking the value one at  $1_G$ .

A positive-definite, continuous, complex-valued function on  $G$  which takes the value one at  $1_G$  is usually called a *state* of  $G$ . We are thus looking for the  $K$ -bi-invariant (and smooth, if need be,) states of  $G$ .

## 2.2 How invariant Gaussian fields correspond to group representations

This subsection describes some results due to Yaglom [35], although the presentation differs slightly because I would like to give direct proofs.

Unitary representations of  $G$  are a natural source of positive-definite functions: if  $U$  is a continuous morphism from  $G$  to the unitary group  $U(\mathcal{H})$  of a Hilbert space  $\mathcal{H}$ , then for every unit vector  $v$  in  $\mathcal{H}$ ,  $g \mapsto \langle v, U(g)v \rangle$  is a state of  $G$ . In fact if  $m$  is a state of  $G$ , there famously is<sup>5</sup> a Hilbert space  $\mathcal{H}_m$  and a continuous morphism from  $G$  to  $U(\mathcal{H}_m)$ , as well

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<sup>5</sup>This is the Gelfand-Naimark-Segal construction: on the vector space  $\mathcal{C}_c(G)$  of continuous, compactly-supported functions on a locally compact second countable unimodular group  $G$ , we can consider the bilinear form  $\langle f, g \rangle_m := \int_G m(x^{-1}y) f(x) \bar{f}(y) dx dy$ . It defines a scalar product on  $\mathcal{C}_c(G) / \{f \in \mathcal{C}_c(G), \langle f, f \rangle_m = 0\}$ , and we can complete this into a Hilbert space  $\mathcal{H}_m$ ; the natural action of  $G$  on  $\mathcal{C}_c(G)$  yields a unitary representation of  $G$  on  $\mathcal{H}_m$ .

as a unit vector  $v_m$  in  $\mathcal{H}_m$ , such that<sup>6</sup>  $m(g) = \langle v_m, U(g)v_m \rangle$ .

The study of  $K$ -bi-invariant states is a classical subject when  $(G, K)$  is a *Gelfand pair*, that is, when  $G$  is connected,  $K$  is compact and the convolution algebra of  $K$ -bi-invariant integrable functions on  $G$  is commutative<sup>7</sup>.

It is immediate from the definition that a state of  $G$  is a bounded function on  $G$ ; thus the  $K$ -bi-invariant states of  $G$  form a convex subset  $\mathcal{C}$  of the vector space  $L^\infty(G)$  of bounded functions. Viewing  $L^\infty(G)$  as the dual of the space  $L^1(G)$  of integrable functions (here we assume a Haar measure is fixed on the – automatically unimodular – group  $G$ ), and equipping it with the weak topology,  $\mathcal{C}$  appears as a relatively compact, convex subset of  $L^\infty(G)$  because of Alaoglu's theorem.

The extreme points of  $\mathcal{C}$  are usually known as *elementary spherical functions for the pair*  $(G, K)$ . Their significance to representation theory is that they correspond to *irreducible* unitary representations: if  $m$  is a state of  $G$  and  $(\mathcal{H}, U, v)$  is such that  $m = g \mapsto \langle v, U(g)v \rangle$  as above, then  $m$  is an elementary spherical function for  $(G, K)$  if and only if the unitary representation  $U$  of  $G$  on  $\mathcal{H}$  is irreducible<sup>8</sup>. The condition of  $K$ -bi-invariance translates into the existence of a  $K$ -fixed vector in  $\mathcal{H}$ .

When  $(G, K)$  is a Gelfand pair, the unitary irreducible representations of  $G$  which have a  $K$ -fixed vector have the subspace of  $K$ -fixed vector one-dimensional and not larger: a consequence is that different elementary spherical functions correspond to nonequivalent class-one<sup>9</sup> representations of  $G$ . So the Gelfand-Naimark-Segal construction yields a bijection between elementary spherical functions for  $(G, K)$  and unitary irreducible representations of  $G$  having a  $K$ -fixed vector.

To come back to the description of general  $K$ -bi-invariant states in terms of the extreme points of  $\mathcal{C}$ , the Choquet-Bishop-de Leeuw representation theorem (a measure-flavoured generalization of the Krein-Milman theorem) exhibits a general  $K$ -bi-invariant state as a "direct integral" of elementary spherical functions, in a way that mirrors the (initially more abstract) decomposition of the corresponding representation of  $G$  into irreducibles. To be precise, let  $\mathbf{\Lambda}$  be the space of extreme points of  $\mathcal{C}$ , a topological space if one lets it inherit the weak topology from  $L^\infty(G)$ . Then Choquet's theorem says every point of  $\mathcal{C}$  is the barycentre of a probability measure concentrated on  $\mathbf{\Lambda}$ , and the probability measure is actually unique in our case: for a discussion and proof see [13], Chapter II.

We can summarize the above discussion with the following statement.

**Proposition 2.2** (the Godement-Bochner theorem). *Suppose  $(G, K)$  is a Gelfand pair, and  $\mathbf{\Lambda}$  is the (topological) space of elementary spherical functions for the pair  $(G, K)$ , or*

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<sup>6</sup>The linear functional  $f \mapsto \int_G f \bar{m}$  extends to a bounded linear functional on  $\mathcal{H}_m$ , and the Riesz representation theorem yields one  $v_m$  in  $\mathcal{H}_m$  which has the desired property.

<sup>7</sup>The subject of positive-definite functions becomes tractable because the Gelfand spectrum of this commutative algebra furnishes a handle on positive-definite functions through the elementary spherical functions to be defined just below.

<sup>8</sup>Indeed, should there be  $U(G)$ -invariant subspaces  $\mathcal{H}_1, \mathcal{H}_2$  such that  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , orthogonal direct sum, writing  $v = v_1 + v_2$  with  $v_i$  in  $\mathcal{H}_i$ , one would have  $m(g) = \langle v_1, U(g)v_1 \rangle + \langle v_2, U(g)v_2 \rangle$ , and  $g \mapsto \langle \frac{v_1}{\sqrt{2}}, U(g) \frac{v_1}{\sqrt{2}} \rangle$  would be a positive-definite function, thus  $m$  would not be an extreme point of  $\mathcal{C}$ . The reverse implication is just as easy using the Gelfand-Naimark-Segal construction.

<sup>9</sup>A class-one representation is an irreducible representation which has nonzero  $K$ -fixed vectors.

equivalently the (topological) space of equivalence classes of unitary irreducible representations of  $G$  having a  $K$ -fixed vector. Then the  $K$ -bi-invariant states of  $G$  are exactly the continuous functions on  $G$  which can be written as  $\varphi = \int_{\Lambda} \varphi_{\lambda} d\mu_{\varphi}(\lambda)$ , where  $\mu_{\varphi}$  is a measure on  $\Lambda$ .

Let us make the backwards way from the theory of positive-definite functions for a Gelfand pair  $(G, K)$  to that of Gaussian random fields on  $G/K$ . It starts with a remark: suppose  $m_1, m_2$  are  $K$ -bi-invariant states of  $G$  and  $\Phi_1, \Phi_2$  are independent Gaussian fields whose covariance functions, when turned into functions on  $G$  as before, are  $m_1$  and  $m_2$ , respectively. Then a Gaussian field whose correlation function is  $m_1 + m_2$  necessarily has the same probability distribution as  $\Phi_1 + \Phi_2$ . A simple application of Fubini's theorem extends this remark to provide a *spectral decomposition for Gaussian fields*, which mirrors the above decomposition of spherical functions:

- For every  $\lambda$  in  $\Lambda$ , there is, up to equality of the probability distributions, exactly one Gaussian field whose correlation function is  $\varphi_{\lambda}$  ;

- Suppose  $(\Phi_{\lambda})_{\lambda \in \Lambda}$  is a collection of mutually independent Gaussian fields, and for each  $\lambda$ ,  $\Phi_{\lambda}$  has correlation function  $\varphi_{\lambda}$ . Then for each probability measure  $\mu$  on  $\Lambda$ , the covariance function of the Gaussian field

$$x \rightsquigarrow \int_{\Lambda} \Phi_{\lambda}(x) d\mu(\lambda)$$

is  $\int_{\Lambda} \varphi_{\lambda} d\mu_{\varphi}(\lambda)$ .

In the next section, I will focus on special cases (most importantly, symmetric spaces); in these cases I will give explicit descriptions of  $\Lambda$  and, for each  $\lambda$  in  $\Lambda$ , of the spherical function  $\varphi_{\lambda}$  and of a Gaussian field whose correlation function is  $\varphi_{\lambda}$ .

### 3 Existence theorems and explicit constructions

#### 3.1 Semidirect products with a vector normal subgroup: easy no-go results

Suppose  $H$  is a Lie group,  $A$  is a finite-dimensional vector space, and  $\rho : H \rightarrow GL(A)$  is a continuous morphism. The semidirect product  $G = H \ltimes_{\rho} A$  (whose underlying set is  $H \times A$ , and whose composition reads  $(h_1, a_1) \cdot (h_2, a_2) := (h_1 h_2, a_1 + \rho(h_1) a_2)$ ) is a Lie group.

Since an  $H$ -bi-invariant function on  $G$  is entirely determined by its restriction to  $A$ , the convolution of  $H$ -bi-invariant functions on  $G$  is a commutative operation. So when  $H$  is compact,  $(G, H)$  is a Gelfand pair; for this case I shall make the situation fully explicit in subsection 3.3 below. When  $H$  is not compact,  $(G, H)$  is not a Gelfand pair, and that is not only because the definition as I wrote it needs the compactness: I shall start this section by showing that  $G$ -invariant continuous Gaussian fields need not exist on  $G/H$ .

**Examples.** The *Poincaré group*  $P$  is the largest subgroup of the affine group of  $\mathbb{R}^4$  under which the space of solutions to Maxwell's (that is, the wave) equation for a scalar-valued field in a vacuum is stable. It is a famous result of Poincaré that  $P = SO(3, 1) \ltimes \mathbb{R}^4$ , with the obvious action of  $SO(3, 1)$  on  $\mathbb{R}^4$ . The *Galilei group* is the subgroup of the affine

group of  $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$  gathering the transformations which, for pairs of points in  $\mathbb{R}^4$ , preserve the notion of Euclidean distance between the ("space") projections on  $\mathbb{R}^3$  as well as the distance between the ("time") projections on  $\mathbb{R}$  (to be formal, the Galilei group consists of affine transformations of  $\mathbb{R}^4$  leaving the map  $[(x_1, t_1), (x_2, t_2)] \mapsto (\|x_1 - x_2\|, t_1 - t_2)$  from  $\mathbb{R}^4 \times \mathbb{R}^4$  to  $\mathbb{R}^2$  invariant<sup>10</sup>).

To state our easy no-go result, recall that the Fourier transform of a function on  $A$  is a function on the set  $\hat{A}$  of characters of  $A$ : . Recall also that an action of  $H$  on the Abelian group  $A$  yields an action  $\hat{\rho}$  of  $H$  on  $\hat{A}$  if we set  $\hat{\rho}(h)\chi := x \mapsto \chi(\rho(h)^{-1}x)$ .

**Proposition 3.1.** *Let  $G$  be a semidirect product  $H \ltimes A$  as above. If there is no compact orbit of  $H$  in  $\hat{A}$  but the trivial one, then no real-valued standard Gaussian field on  $\mathbb{R}^4$  whose probability distribution is  $G$ -invariant can have continuous trajectories.*

*Proof.* Such a field would yield a positive-definite, continuous,  $H$ -bi-invariant function, say  $\tilde{\Gamma}$ , on  $G$ , taking the value one at  $1_G$ , but we shall see now that there can be no such function except the constant one. Write  $\Gamma$  for the restriction of  $\tilde{\Gamma}$  to  $A$ ; because  $\Gamma$  is positive-definite as a function on the abelian group  $A$ , Bochner's theorem (see for instance [?]) says it is the Fourier transform of a bounded measure  $\nu_\Gamma$  on  $\hat{A}$ . Because  $\Gamma$  is invariant under the linear action on  $H$  on  $A$  and because of the elementary properties of the Fourier transform, the measure  $\nu_\Gamma$  must also be invariant, and so if  $\Omega$  is a compact subset of an  $H$ -orbit in  $\hat{A}$ ,  $\nu_\Gamma(\Omega)$  must be equal to  $\nu_\Gamma(h \cdot \Omega)$  for each  $h$  in  $H$ . That is not possible when  $H \cdot \Omega$  is noncompact unless  $\nu_\Gamma(\Omega)$  is zero, because there is a sequence  $(h_n)$  in  $H^\mathbb{N}$  such that  $\cup_n h_n \cdot \Omega$  is a disjoint union, and because the total mass of  $\nu_\Gamma$  is finite. A consequence is that the support of  $\nu_\Gamma$  must be the origin in  $\hat{A}$ , and since  $\nu_\Gamma$  is the Fourier transform of a continuous function, it must be a multiple of the Dirac mass at zero.  $\square$

If  $H \ltimes A = SO(3, 1) \ltimes \mathbb{R}^4$  is the Poincaré group, we can identify  $\hat{A}$  with  $\mathbb{R}^4$  in a  $H$ -equivariant way using Minkowski's quadratic form, and then the orbits of  $H$  on  $\hat{A}$  appear as the level sets of Minkowski's quadratic form in  $\mathbb{R}^4$ . So of course the hypothesis of Proposition 3.1 is satisfied:

**Corollary.** *No standard real-valued Gaussian field on  $\mathbb{R}^4$  whose probability distribution is invariant under the Poincaré group can have continuous trajectories.*

Let us now consider whether a standard real-valued Gaussian field on  $\mathbb{R}^4$  whose probability distribution is invariant under the Galilei group can have continuous trajectories.

The linear part of an element in the Galilei group reads  $(x, t) \mapsto (A\vec{x} + \vec{v}t, t)$ , where  $A$  is an element of  $SO(3)$  and  $\vec{v}$  is a vector in  $\mathbb{R}^3$ , and its inverse reads  $(x, t) \mapsto (A^{-1}\vec{x} - (A^{-1}\vec{v})t, t)$ ; so if  $\chi$  is in  $\widehat{\mathbb{R}^4}$  and decomposes as  $(\vec{x}, t) \mapsto \langle \vec{k}_\chi, \vec{x} \rangle + \omega_\chi t$ , then  $\chi(A^{-1}\vec{x} - (A^{-1}\vec{v})t) = \langle A\vec{k}_\chi, \vec{x} \rangle + (\omega_\chi + \langle \vec{k}_\chi, \vec{v} \rangle)t$ . This means that if  $h$  is the element  $(x, t) \mapsto (A\vec{x} + \vec{v}t, t)$  of the linear part of the Galilei group,

$$h \cdot \chi := h \cdot (\vec{k}_\chi, \omega_\chi) = (A\vec{k}_\chi, \omega_\chi + \langle \vec{k}_\chi, \vec{v} \rangle).$$

The orbits of the Galilei group on  $\widehat{\mathbb{R}^4}$  are thus the cylinders  $C_\kappa := \{(\vec{k}, \omega) \mid \|\vec{k}\| = \kappa\}$ ,  $\kappa > 0$ , and the points  $\{(0, \omega)\}$ ,  $\omega \in \mathbb{R}$ . The proof of Proposition 3.1 shows that the support of  $\nu_\Gamma$  must be the union of the compact orbits, and this is the "time frequency" axis

<sup>10</sup>The action of the affine group here is the diagonal action on  $\mathbb{R}^4 \times \mathbb{R}^4$ .



$\{(0, \omega) \mid \omega \in \mathbb{R}\}$ ; the measure  $\nu_{\Gamma}$  then appears as the product of the Dirac mass on the line of  $\mathbb{R}^4$  which is dual to the “time” axis, with a bounded measure on that line.

So a standard real-valued Gaussian field on  $\mathbb{R}^4$  whose probability distribution is invariant under the Galilei group cannot have continuous trajectories without losing any form of space dependence :

**Corollary.** *A standard, real-valued Gaussian field whose probability distribution is invariant under the Galilei group and which has continuous trajectories reads but  $(x, t) \mapsto \Phi(t)$ , where  $\Phi$  is a stationary and continuous Gaussian field on the real line.*

**Remark 3.1.** Proposition 3.1 and its corollaries might seem incompatible with the fact that, leaving Gelfand pairs aside, every unitary representation contributes a continuous positive-definite function. The representation-theoretic counterpart to Proposition 3.1 is thus the fact that no irreducible unitary representation of  $H \ltimes A$  except the trivial one can have a  $H$ -invariant vector.

If we look for fields with *smooth* trajectories instead of continuous ones<sup>11</sup>, an interesting remark by Adler and Taylor makes Proposition 3.1 trivial:

**Lemma 3.1.** *If there exists a smooth, non-constant, real-valued Gaussian field on  $G/K$  whose probability distribution is  $G$ -invariant, then  $K$  is compact.*

*Proof.* For each  $p$  in  $G/K$  and every  $(X_p, Y_p)$  in  $(T_p(G/K))^2$ , set

$$g(X_p, Y_p) := \mathbb{E}[(d\Phi(p)X_p)(d\Phi(p)Y_p)].$$

This has a meaning as soon as the samples of  $\Phi$  are almost surely smooth, and it does define a riemannian metric on  $G/K$ . The invariance of the field now implies that this metric is  $G$ -invariant, and in particular that the positive-definite quadratic form it provides on  $T_{x_0}(G/K)$  is  $K$ -invariant. Thus  $K$  is contained in the isometry group of a finite-dimensional Euclidean space, so it is compact.  $\square$

### 3.2 Monochromatic fields on commutative spaces

Let us start again with a Gelfand pair  $(G, K)$  with connected  $G$ . From now on, I shall assume that  $G$  is a *Lie* group and focus on Gaussian fields which have *smooth* trajectories. The reason, here summarized as Theorem 3.1, is that in this case, the spherical functions are solutions to invariant partial differential equations: as I promised earlier, the coefficients for these partial differential equations must determine all the statistical properties the corresponding field, and we shall see this at work with the density of the zero-set. In addition, fully explicit constructions are possible in many cases of interest.

A good reference for this subsection is J. A. Wolf [33].

Let me write  $\mathcal{D}_G(X)$  for the algebra of  $G$ -invariant differential operators on  $X = G/K$ . Then Thomas [28] and Helgason [16] proved that  $(G, K)$  is a Gelfand pair if and only if

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<sup>11</sup>In another direction, one could argue that the inverse Fourier transform of a noncompact orbit, while not a continuous spherical function, is a tempered distribution which could be used to define distribution-valued Gaussian fields on  $\mathbb{R}^4$ ; although we shall not take up this point of view, Yaglom mentioned the possibility at the end of [35], and on the group-theoretic side, the distribution-theoretic theory of *generalized* Gelfand pairs has been worked out, see [11].

$\mathcal{D}_G(X)$  is a commutative algebra.

**Theorem** (See [33], Theorems 8.3.3-8.3.4). *A smooth,  $K$ -bi-invariant function  $\phi$  is an elementary spherical function for  $(G, K)$  if and only if there is, for each  $D$  in  $\mathcal{D}_G(X)$ , a complex number  $\chi(D)$  such that*

$$D\phi = \chi_\phi(D)\phi.$$

*The eigenvalue assignment  $D \mapsto \chi_\phi(D)$  defines a character of the commutative algebra  $\mathcal{D}_G(X)$ . It determines the spherical function  $\phi$ : when  $\chi$  is a character of  $\mathcal{D}_G(X)$ , there is a unique spherical function  $\phi$  such that  $\chi = \chi_\phi$ .*

**Definition 3.2.** A standard Gaussian random field on  $X$  whose correlation function is a multiple of an elementary spherical function will be called *monochromatic*; the corresponding character of  $\mathcal{D}_G(X)$  will be called its *spectral parameter*.

Note that with a choice of  $G$ -invariant riemannian metric on  $G/K$  comes an element of  $\mathcal{D}_G(X)$ , the Laplace-Beltrami operator  $\Delta_X$ .

**Example 3.3.** If  $X$  is a *two-point homogeneous space*, that is, if  $G$  is transitive on equidistant pairs<sup>12</sup> of points in  $X$ , then  $\mathcal{D}_G(X)$  is the algebra of polynomials in  $\Delta_X$ .

**Example 3.4.** If  $X$  is a *symmetric space* (see below), then  $\mathcal{D}_G(X)$  is finitely generated; thus a character of  $\mathcal{D}_G(X)$  is specified by a finite collection of real numbers.

### 3.3 Explicit constructions. A: Flat homogeneous spaces

Suppose  $(G, K)$  is a Gelfand pair, and the commutative space  $X = G/K$  is *flat*. Then we know from early work by J. A. Wolf (see [34], section 2.7) that  $X$  is isometric to a product  $\mathbb{R}^n \times T^s$ .

I will be concerned with the simply connected case: let  $V$  be a Euclidean space,  $K$  be a closed subgroup of  $SO(V)$ , and  $G$  be the semidirect product  $K \ltimes V$ . Then as we saw  $(G, K)$  is a Gelfand pair, and we can describe the  $G$ -invariant continuous Gaussian fields on  $V = G/K$  from the monochromatic ones.

The next proposition provides a description of the elementary spherical functions.

**Proposition 3.2.** *Suppose  $V$  is a Euclidean vector space, and  $K$  is a closed subgroup of  $SO(V)$ . Then the Fourier transform of a  $K$ -orbit in  $V$  is a smooth function; once normalized to take the value one at zero, it is an elementary spherical function for the Gelfand pair  $(K \ltimes V, K)$ . In fact, every elementary spherical function for  $(K \ltimes V, K)$  restricts on  $V$  to the Fourier transform of a  $K$ -orbit in  $V$ .*

*Proof.* To get a handle on the  $K$ -bi-invariant states of  $K \ltimes V$ , let me start with a bounded measure on the orbit space  $V/K$  and the measure  $\tilde{\mu}$  on  $V$  obtained by pulling  $\mu$  back with the help of the Hausdorff measure of each  $K$ -orbit (normalized so that each orbit

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<sup>12</sup>This means that two pairs of points  $(p_1, q_1)$ ,  $(p_2, q_2)$  satisfy  $d(p_1, p_2) = d(q_1, q_2)$  if and only if there is an isometry  $g \in G$  such that  $g \cdot p_1 = q_1$  and  $g \cdot p_2 = q_2$ .

has total mass one). I first remark that the Fourier transform of  $\tilde{\mu}$  provides a positive-definite function for  $(K \ltimes V, K)$ . Indeed, Bochner's theorem says it provides a positive-definite function on  $V$ , and if  $(k_1, v_1)$  and  $(k_2, v_2)$  are elements of  $K \ltimes V$ ,  $(k_1, v_1)(k_2, v_2)^{-1}$  is equal to  $(k_1 k_2^{-1}, v_1 - k_1 k_2^{-1} v_2)$ , so a  $K$ -bi-invariant function takes the same value at  $(k_1, v_1)(k_2, v_2)^{-1}$  as it does at  $(k_1^{-1}, 0)(k_1, v_1)(k_2, v_2)^{-1}(k_2, 0) = (1_K, k_1^{-1} v_1 - k_2^{-1} v_2)$ ; this checks the positive-definiteness directly.

Now suppose  $\tilde{\Gamma}$  is a  $K$ -bi-invariant state of  $K \ltimes V$ , and write  $\Gamma$  for its restriction to  $V$ , a bounded  $K$ -invariant positive-definite continuous function on  $V$ . The Fourier transform of  $\Gamma$ , a bounded complex measure on  $V$  with total mass one because of Bochner's theorem, is  $K$ -invariant, and yields a bounded measure on the orbit space  $K/V$ . If the support of this measure is not a singleton, we can split it as the half-sum of bounded measures with total mass one, and lifting them to  $V$  and taking Fourier transform exhibits our initial state as a sum of two  $K$ -bi-invariant functions taking the value one at  $1_{K \ltimes V}$  which, according to the previous paragraph, are positive-definite. So the extreme points among the  $K$ -bi-invariant states correspond to  $K$ -invariant measures concentrated on a single  $K$ -orbit in  $V$ , which proves the proposition.  $\square$

**Remark 3.5.** Suppose  $\mathcal{D}_G(V)$  is finitely generated. Since the elements in  $\mathcal{D}_G(V)$  are invariant under the translations of  $V$ , they have constant coefficients, so they become multiplication by polynomials after taking Fourier transform. Taking the Fourier transform of what Proposition 3.1 says, we see that a  $K$ -orbit in  $V$  is an affine algebraic subset of  $V$ , and that all orbits are obtained by varying the constant terms in a generating system for the ring of Fourier transforms of elements of  $\mathcal{D}_G(V)$ . Of course the simplest case is when the orbits are spheres and  $G$  is the Euclidean motion group of  $V$ .

Proposition 3.2 is explicit enough to allow for computer simulation: suppose  $\varphi$  is an elementary spherical function, and let us see how to build a Gaussian field  $\Phi_\Omega$  on  $V$  whose covariance function is  $\varphi$ . By definition, we must have  $\mathbb{E}[\Phi_\Omega(x)\Phi_\Omega(0)] = \varphi(x)$ , so using Fubini's theorem we see that (almost) all samples of  $\Phi$  must have their Fourier transform concentrated on the same  $K$ -orbit of  $V$ , say  $\Omega$ , as  $\Phi$ . Thus  $\Phi$  is a random superposition of waves whose wave-vectors lie on  $\Omega$ .

**Lemma 3.2.** *Assume  $(\zeta_{\vec{k}})_{\vec{k} \in \Omega}$  is a collection of mutually independent standard Gaussian random variables. Normalize the Hausdorff measure on  $\Omega$  so that it has total mass one. Then the Gaussian random field*

$$\Phi_\Omega := x \mapsto \left[ \int_\Omega e^{i\vec{x} \cdot \vec{k}} \zeta_{\vec{k}} d\vec{k} \right].$$

*is  $G$ -homogeneous, smooth, and has covariance function  $\varphi_\Omega$ .*

*Proof.* This is straightforward from the definition, since applying Fubini's theorem twice

yields

$$\begin{aligned}
\mathbb{E}[\Phi_\Omega(\vec{x})\Phi_\Omega(0)^*] &= \mathbb{E}\left[\left(\int_\Omega e^{i\vec{k}\cdot\vec{x}}\zeta_{\vec{k}}d\vec{k}\right)\left(\int_\Omega \zeta_{\vec{u}}^*d\vec{u}\right)\right] \\
&= \mathbb{E}\left[\int_{\Omega^2} e^{i\vec{k}\cdot\vec{x}}\zeta_{\vec{k}}\zeta_{\vec{u}}^*d\vec{k}d\vec{u}\right] \\
&= \int_{\Omega^2} e^{i\vec{k}\cdot\vec{x}}\mathbb{E}[\zeta_{\vec{k}}\zeta_{\vec{u}}^*]d\vec{k}d\vec{u} \\
&= \int_\Omega e^{i\vec{k}\cdot\vec{x}}d\vec{k} \\
&= \varphi_\Omega(\vec{x})
\end{aligned}$$

as announced. The smoothness and invariance follow from Proposition 2.1.  $\square$

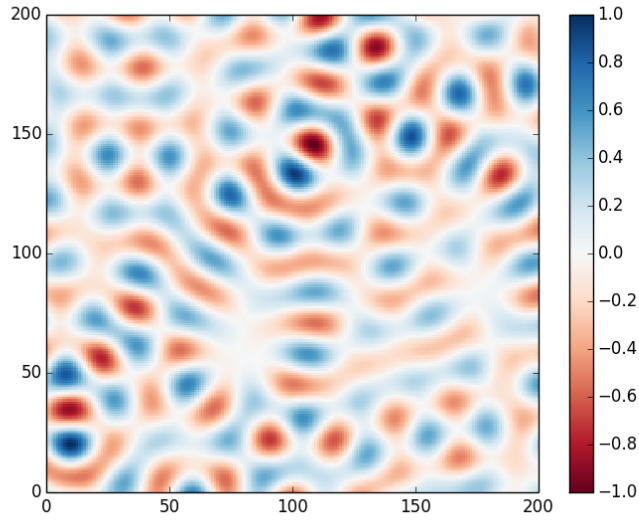


Figure 1: a real-valued map, sampled from a real-valued monochromatic field on the Euclidean plane.

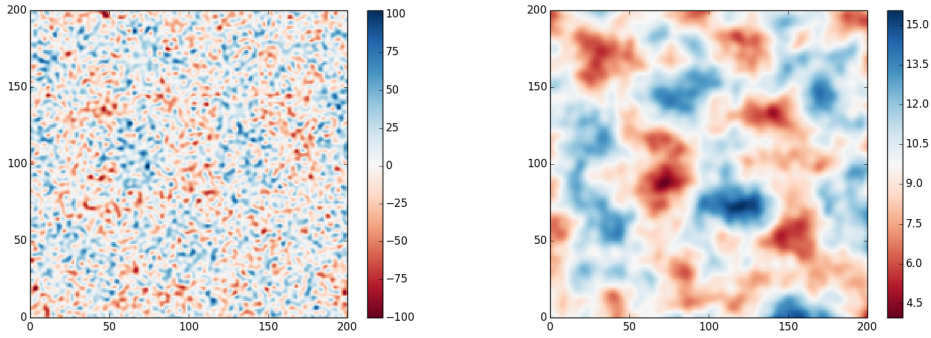


Figure 2: Two real-valued maps, sampled from real-valued invariant fields on the Euclidean plane: because of Proposition 3.2, the elementary spherical functions form a half-line; the power spectrum (the measure on  $\mathbb{R}^+$  defined in Proposition 2.2) of the upper map is roughly the indicatrix of a segment, the power spectrum of the lower one has the same support but decreases as  $1/R^2$

### 3.4 Explicit constructions. B: Compact homogeneous spaces

Suppose  $(G, K)$  is a Gelfand pair, and the commutative space  $X = G/K$  is *positively curved*. Then  $G$  is a connected *compact* Lie group, and the Hilbert spaces for irreducible representations of  $G$  are finite-dimensional.

A consequence is that if  $T : G \rightarrow U(\mathcal{H})$  is an irreducible representation, the map  $g \mapsto \text{Trace}(T(g))$  is a continuous, complex-valued function; it is of course the *global character* of  $G$ .

**Proposition 3.3** (G. Van Dijk, see Theorem 6.5.1 in [11]). *The elementary spherical functions for  $(G, K)$  are the maps*

$$x \mapsto \int_K \chi(x^{-1}k) dk$$

where  $\chi$  runs through the set of global characters of irreducible representations of  $G$  having a  $K$ -fixed vector, and the integration is performed w.r.t the normalized Haar measure of  $K$ .

Note that if  $\chi$  is the global character of an irreducible representation of  $G$  which has no  $K$ -fixed vector, the above expression is zero.

The reason why this provides an explicit formula for the spherical functions is that Hermann Weyl famously wrote down the global character of an irreducible representation of  $G$ . Let  $T$  be a maximal torus in  $G$ , let  $\mathfrak{t}$  and  $\mathfrak{g}$  be the complexified Lie algebras of  $T$  and  $G$ , and let  $W$  be the Weyl group of the pair  $(\mathfrak{g}, \mathfrak{t})$ ,  $\mathcal{C} \subset \mathfrak{t}^*$  be a Weyl chamber in  $\mathfrak{t}^*$ ,  $\Sigma$  be the set of positive roots of  $(\mathfrak{g}, \mathfrak{t})$  in the ordering determined by  $\mathcal{C}$  – a subset of  $\mathfrak{t}^*$  as well –,  $\rho$  be the half-sum of elements of  $\Sigma$ ,  $\Lambda$  be the subset of  $\mathfrak{t}^*$  gathering the differentials of continuous morphisms  $T \rightarrow \mathbb{C}$ , and  $\Lambda^+$  be  $\Lambda \cap \mathcal{C}$ . Two of the most famous results in representation theory are:

- There is a natural bijection (the highest-weight theory) between  $\Lambda^+$  and the equivalence classes of irreducible representation of  $G$ ;
- The global character of all irreducible representations with highest weight  $\lambda$  restricts to  $\exp_G(\mathfrak{t})$  as<sup>13</sup>

$$e^H \mapsto \frac{\sum_{w \in W} \varepsilon(w) e^{\langle \lambda + \rho, wH \rangle}}{\sum_{w \in W} \varepsilon(w) e^{\langle \rho, wH \rangle}}.$$

This gives a completely explicit description of the covariance functions of invariant Gaussian random fields on  $X$  (provided one can find a maximal torus, the Weyl group, the roots... explicitly: the **atlas** software seems to do that – and much more – when  $G$  is reductive). In contrast to what happened above for flat spaces and to what will happen below for symmetric spaces, however, I am not aware that this leads to an explicit description of the Gaussian random field with a given spherical function as its covariance function. We must stick to Yaglom's general construction here: without assuming that  $(G, K)$  is a Gelfand pair but only that it is a pair of connected compact Lie groups, let  $T : G \rightarrow U(\mathcal{H})$  be an irreducible representation,  $(\mathbf{e}_1, \dots, \mathbf{e}_r)$  be an orthonormal basis for the space of  $K$ -fixed vectors, and let  $(\mathbf{e}_{r+1}, \dots, \mathbf{e}_d)$  be an orthonormal basis for its orthocomplement. Yaglom proved the following two facts:

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<sup>13</sup>Recall that the conjugates of  $\exp_G(\mathfrak{t})$  is  $G$  and the character is conjugation-invariant!

- The maps  $gK \mapsto \langle \mathbf{e}_i, T(g)\mathbf{e}_j \rangle$ ,  $i, j = 1..r$ , are elementary spherical functions for  $(G, K)$ ,
- If  $(\zeta_i)_{i=1, \dots, d}$  is a collection of i.i.d. standard Gaussian variables, then for every  $i_0$  in  $\{1, \dots, r\}$ ,

$$gK \mapsto \sum_{i=1}^d \zeta_i \langle \mathbf{e}_i, T(g)\mathbf{e}_{i_0} \rangle \quad (1)$$

is an invariant standard Gaussian random field on  $G/K$ , whose covariance function is  $gK \mapsto \langle \mathbf{e}_1, T(g)\mathbf{e}_1 \rangle$ .

While Yaglom's result is rather abstract compared to the above descriptions for flat spaces, in many cases of interest explicit bases  $(\mathbf{e}_i)$  and explicit formulae for the matrix elements  $\langle \mathbf{e}_i, T(g)\mathbf{e}_j \rangle$  are known (the obvious reference is [29]), making (1) startlingly concrete.

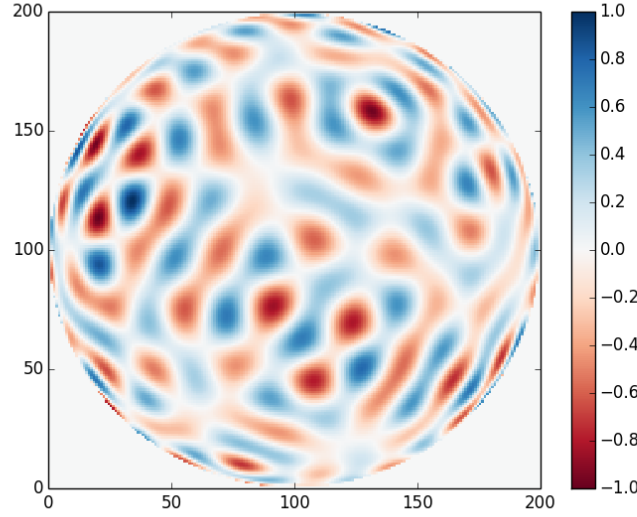


Figure 3: a sample from a real-valued monochromatic field on the sphere. This uses a combination of spherical harmonics with i.i.d. gaussian coefficients.

### 3.5 Explicit constructions. C: Symmetric spaces of noncompact type

Suppose  $(G, K)$  is a Gelfand pair, and the commutative space  $X = G/K$  is *negatively curved*. Then  $G$  is noncompact, and without any additional hypothesis on  $G$  it is quite difficult to do geometry and analysis on  $X$ . It is easier to do so if  $X$  is a *symmetric* space. The isometry group  $G$  is then semisimple.

In that case Harish-Chandra determined the elementary spherical functions for  $(G, K)$  in 1958; Helgason later reformulated his discovery in a way which brings it very close to Proposition 3.2. For the contents of this subsection, see chapter III in [17], and see of course [15], [16], [17] for more on the subject.

Suppose  $G$  is a connected semisimple Lie group with finite center and  $K$  is a maximal compact subgroup in  $G$ . Write  $\mathfrak{g}$  and  $\mathfrak{k}$  for their Lie algebras,  $\mathfrak{p}$  for the orthocomplement of  $\mathfrak{k}$  with respect to the Killing form of  $\mathfrak{g}$ ,  $\mathfrak{a}$  for a maximal abelian subspace of  $\mathfrak{p}$ . Using

a subscript  $\cdot_{\mathbb{C}}$  to denote complexifications, let  $\mathcal{C} \subset i\mathfrak{a}^* \subset \mathfrak{a}_{\mathbb{C}}^*$  be the Weyl chamber corresponding to a choice of positive roots for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ , and  $\rho$  be the corresponding half-sum of positive roots. The direct sum of real root spaces for the chosen positive roots is a Lie subalgebra, say  $\mathfrak{n}$ , of  $\mathfrak{g}$ , and if  $A$  and  $N$  are the subgroups  $\exp_G(\mathfrak{a})$  and  $\exp_G(\mathfrak{n})$  of  $G$ , the map  $(k, a, n) \mapsto kan$  is a diffeomorphism between  $K \times A \times N$  and  $G$ . When the Iwasawa decomposition of  $x \in G$  accordingly is  $k \exp_G(H)n$ , let me write  $\mathfrak{A}(x) = H$  for the  $\mathfrak{a}$ -component.

Now suppose  $\lambda$  is in  $\mathfrak{a}^*$  and  $b$  is in  $K$ . Define

$$e_{\lambda, b} : G \rightarrow \mathbb{R} \\ x \mapsto e^{(i\lambda + \rho \mid \mathfrak{A}(b^{-1}x))}.$$

Then  $e_{\lambda, b}$  defines a smooth function from  $X = G/K$  to  $\mathbb{C}$ ; it is an eigenfunction of  $\Delta_X$ , with eigenvalue<sup>14</sup>  $-(\|\lambda\|^2 + \|\rho\|^2)$ . These functions are useful for harmonic analysis on  $G/K$  in about the same way as plane waves are useful for classical Fourier analysis.

If  $(\lambda_1, b_1)$  and  $(\lambda_2, b_2)$  are elements of  $\mathfrak{a}^* \times K$ , then  $e_{\lambda_1, b_1}$  and  $e_{\lambda_2, b_2}$  coincide if and only if there is an element  $w$  in the Weyl group of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$  such that  $\lambda_1 = w\lambda_2$  and if  $b_1$  and  $b_2$  have the same image in the quotient  $B = K/M$ , where  $M$  is the centralizer of  $\mathfrak{a}$  in  $K$ . Each of the  $e_{\lambda, b}$  thus coincides with exactly one of the  $e_{\lambda^+, b_s}$ , where  $\lambda^+$  runs through the closure  $\Lambda^+$  of  $\mathcal{C}$  in  $i\mathfrak{a}^*$ .

**Theorem** (Harish-Chandra). *For each  $\lambda$  in  $\Lambda^+$ ,*

$$\varphi_{\lambda} := x \mapsto \int_B e_{\lambda, b}(x) db$$

*is<sup>15</sup> an elementary spherical function for  $(G, K)$ . Every spherical function for  $(G, K)$  is one of the  $\varphi_{\lambda}$ ,  $\lambda \in \Lambda^+$ .*

Thus the possible spectral parameters for monochromatic fields occupy a closed cone  $\Lambda^+$  in the Euclidean space  $i\mathfrak{a}^*$  (and the topology on the space of spherical functions described in section 2.2 coincides with the topology inherited from  $\mathfrak{a}^*$ ). The fact that spherical functions here again appear as a constructive interference of waves yields an explicit description for the monochromatic field with spectral parameter  $\lambda$  (same proof as Lemma 3.2):

**Lemma 3.3.** *Assume  $(\zeta_b)_{b \in B}$  is a collection of mutually independent standard Gaussian random variables. Then the Gaussian field*

$$\Phi_{\lambda} := x \mapsto \left[ \int_B e_{\lambda, b} \zeta_b db \right].$$

*is  $G$ -homogeneous, smooth, and has covariance function  $\varphi_{\lambda}$ .*

<sup>14</sup>Here the norm is the one induced by the Killing form.

<sup>15</sup>Here the invariant measure on  $B$  is normalized so as to have total mass one.

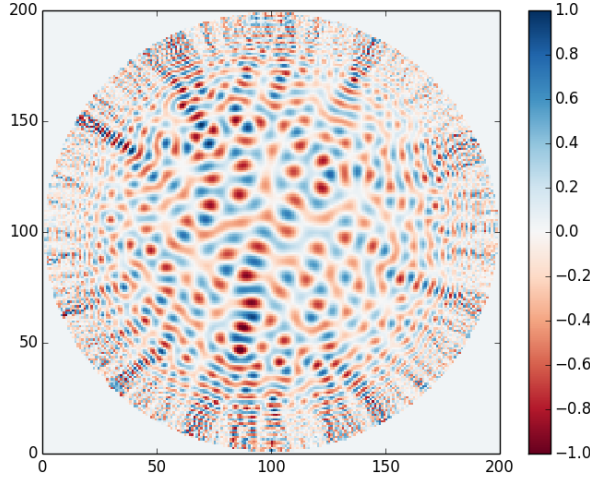


Figure 4: a real-valued map on the Poincaré disk, sampled from a monochromatic field using Lemma 3.3.

## 4 The typical spacing in an invariant field

Let us start with a homogeneous *real*-valued Gaussian field  $\Phi$  on a riemannian homogeneous space  $X$  with isometry group  $G$ . In view of the above pictures, if the correlation function of  $\Phi$  is close enough to being an elementary spherical function, one expects  $\Phi$  to exhibit some form of quasiperiodicity<sup>16</sup>.

Let us now see whether we can give a meaning to the “quasiperiod”. Draw a geodesic  $\gamma$  on  $X$ , and if  $\Sigma$  is a segment on  $\gamma$ , write  $\mathcal{N}_\Sigma$  for the random variable recording the number of zeroes of  $\Phi$  on  $\Sigma$ . Because the field  $\Phi$  is homogeneous and the metric on  $X$  is invariant, the probability distribution of  $\mathcal{N}_\Sigma$  depends only on the length, say  $|\Sigma(\gamma)|$ , of  $\Sigma$ . The identity component of the subgroup of  $G$  fixing  $\gamma$  is a one-parameter subgroup of  $G$ , and reads  $\exp_G(\mathbb{R}\tilde{\gamma})$  for some  $\tilde{\gamma}$  in  $\mathfrak{g}$ ; it is isomorphic to a circle if  $X$  is of the compact type, and isomorphic to the additive group of the real line if  $X$  is of the Euclidian or noncompact type. In any case, this means we can pull back  $\Phi|_\gamma$  to  $\mathbb{R}\tilde{\gamma}$  and view it as a stationary, real-valued Gaussian field on the real line. In this way, the group exponential relating  $\mathbb{R}\tilde{\gamma}$  to  $\gamma$  sends the Lebesgue measure of  $\mathbb{R}$  to a constant multiple of the metric  $\gamma$  inherits from that of  $X$ . The zeroes of the pullback of  $\Phi|_\gamma$  to  $\mathbb{R}\tilde{\gamma}$  can thus be studied through the classical, one-dimensional, Kac-Rice formula:

**Proposition** (Rice’s formula). *Suppose  $\Phi$  is a translation-invariant smooth Gaussian field on the real line, with smooth trajectories; choose a real number  $u$ , and consider an interval  $I$  of length  $\ell$  on the real line. Write  $\mathcal{N}_{u,I}$  for the random variable recording the number of points  $x$  on  $I$  where  $\Phi(x) = u$ ; then*

$$\mathbb{E}[\mathcal{N}_{u,I}] = \ell \cdot \frac{e^{-u^2/2} \sqrt{\lambda}}{\pi} \quad (2)$$

where  $\lambda = \mathbb{E}[\Phi'(0)^2]$  is the second spectral moment of the field.

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<sup>16</sup>In a mathematically loose sense.



An immediate consequence of (2) is that the expectation  $\mathbb{E}[\mathcal{N}_\Sigma]$  depends linearly on  $|\Sigma|$ .

**Definition 4.1.** The *typical spacing* of  $\Phi$  is the positive number  $\Lambda(\Phi)$  such that

$$\frac{1}{\Lambda(\Phi)} := \frac{\mathbb{E}[\mathcal{N}_\Sigma]}{|\Sigma(\gamma)|}.$$

For a comment on the definition, see Example 4.2 below.

**Proposition 4.1.** Suppose  $X$  is a riemannian homogeneous space, and in the setting of Definition 5.1, assume the samples of  $\Phi$  lie almost surely in the eigenspace  $\{f \in C^\infty(X) \mid \Delta_X f = K f\}$ , and write  $\beta$  for the variance of  $\Phi(x)$  at any point  $x \in X$ . Then

$$\Lambda(\Phi) = \frac{\pi}{\sqrt{(\dim X)\beta|K|}}.$$

*Proof.* Let me write  $\kappa$  for the second spectral moment of the stationary gaussian field on the real line, say  $\mathbf{u}$ , obtained by restricting  $\Phi_\gamma$  to  $\mathbb{R}_{\tilde{\gamma}}$  as above:  $\kappa$  is the variance  $\mathbb{E}[\mathbf{u}'(0)^2]$ . Because of (2),  $\Lambda(\Phi)$  is equal to  $\frac{\pi}{\sqrt{\kappa}}$ .

Now,  $\mathbf{u}'(0)$  is the derivative of  $\Phi$  in the direction  $\tilde{\gamma}$ . Its variance can be recovered from the second derivative of the covariance function of  $\Phi$  in the direction  $\tilde{\gamma}$ : let me write  $\Gamma$  for the covariance function of  $\Phi$ , turned into a function on  $G$  thanks to a choice of base point  $x_0$  in  $X$ . Recall that  $\Gamma(a^{-1}b) = \mathbb{E}[\Phi(a \cdot x_0)\Phi(b \cdot x_0)]$ , consider the functions  $f_1 : (a, b) \rightarrow \Gamma(a^{-1}b)$  and  $f_2 : (a, b) \rightarrow \mathbb{E}[\Phi(a \cdot x_0)\Phi(b \cdot x_0)]$  from  $G^2$  to  $\mathbb{C}$ . Write the Lie derivative in the direction  $\tilde{\gamma}$  with respect to  $a$  or  $b$  as  $L_{\tilde{\gamma}}^a$  or  $L_{\tilde{\gamma}}^b$ . Then  $(L_{\tilde{\gamma}}^a L_{\tilde{\gamma}}^b f_1)(1_G) = -L_{\tilde{\gamma}}^2(\Gamma)(1_G)$ , while  $(L_{\tilde{\gamma}}^a L_{\tilde{\gamma}}^b f_2)(1_G) = \mathbb{E}[(L_{\tilde{\gamma}}\Phi)(x_0)^2]$ . Naturally  $f_1 = f_2$ , so

$$\mathbb{E}[(L_{\tilde{\gamma}}\Phi)(x_0)^2] = -L_{\tilde{\gamma}}^2(\Gamma)(1_G).$$

If  $\Gamma_X$  is the map  $x \mapsto \Gamma(x)$ , then  $-L_{\tilde{\gamma}}^2(\Gamma_X)(x_0)$ . Of course, the Laplace-Beltrami operator on  $X$  has much to do with second derivatives :

- when  $X$  is flat,  $\Delta_X$  is the usual laplacian, we can choose Euclidean coordinates on  $X$  such that  $\mathbb{R}_{\tilde{\gamma}}$  is the first coordinate axis; writing  $X_i$  for the vector fields generating the translations along the coordinate axes, we then have  $\Delta_X = \sum_{i=1}^{(\dim X)} L_{X_i}^2$ .

- In the general case, we can localize the computation and use normal coordinates around  $x_0$ : suppose  $(\gamma_1^{x_0}, \dots, \gamma_p^{x_0})$  is an orthonormal basis of  $T_{x_0}X$ , and let  $\tilde{\gamma}_1, \dots, \tilde{\gamma}_p$  be elements of  $\mathfrak{g}$  whose induced vector fields on  $X$  coincide at  $x_0$  with the  $\gamma_i$ s. Then  $(\Delta_X \Gamma)(x_0) = \sum_{i=1}^p (L_{\tilde{\gamma}_i} \Gamma)(x_0)$ .

We now use the fact that the field is  $G$ -invariant and note that the directional derivatives of  $\Gamma_X$  at the identity coset are all identical ; so

$$(L_{\tilde{\gamma}}^2 \Gamma)(x_0) = \frac{1}{\dim X} (\Delta_X \Gamma_X)(x_0).$$

In the special case where  $\Gamma$  is an eigenfunction of  $\Delta_X$ , we thus get

$$\kappa = (\dim X)|K|\Gamma(0) = (\dim X)\beta|K|$$

(recall that  $K$  is nonpositive when  $X$  is compact and nonnegative otherwise), and Proposition 4.1 follows.  $\square$

**Example 4.2.** Suppose  $X$  is the Euclidean plane, and we start from the monochromatic complex-valued invariant field, say  $\Phi$ , with characteristic wavelength  $\lambda$ . Then its real part  $\Phi_{\mathbb{R}}$  has  $\beta = 1/2$  and  $\Lambda(\Phi_{\mathbb{R}}) = \lambda$ . This we may have expected, since the samples of  $\Phi$  are superpositions of waves with wavelength  $\lambda$ .

When the curvature is nonzero, however, Proposition 5.1 seems to say something non-trivial.

**Example 4.3.** Suppose  $X$  is a symmetric space of noncompact type, and we start from a monochromatic invariant field, say  $\Phi$ , with spectral parameter  $\omega$  and  $\beta = 1/(\dim X)$ . In the notations of section 3.4, we get

$$\Lambda(\Phi) = \frac{2\pi}{\sqrt{|\omega|^2 + |\rho|^2}}.$$

This is not quite as unsurprising as Example 4.1 : the samples of  $\Phi$  are superpositions of Helgason waves whose phase surfaces line up at invariant distance  $\frac{2\pi}{|\omega|}$ . The curvature-induced shift in the typical spacing comes from the curvature-induced growth factor in the eigenfunctions for  $\Delta_X$ .

**Example 4.4.** Suppose  $X$  is a compact homogeneous space. Then the gap between zero and the first nonzero eigenvalue<sup>17</sup> of  $\Delta_X$  provides a nontrivial upper bound for the typical spacing of invariant gaussian fields on  $X$  (this upper bound is *not* the diameter of  $X$ ).

This is clear from Lemma 4.1 for fields with samples in an eigenspace of  $\Delta_X$ , and the next lemma will make it clear for other fields also.

For general invariant fields on commutative spaces, we can recover the typical spacing as follows:

**Lemma 4.1.** *Suppose  $X$  is a commutative space,  $\Phi$  is a smooth, invariant, real-valued Gaussian field on  $X$ , and write  $\beta$  for the variance of  $\Phi(x)$  at any point  $x \in X$ . Write the spectral decomposition of  $\Phi$  (section 2) as*

$$\Phi = \int_{\Lambda} \Phi_{\lambda} dP(\lambda);$$

then

$$\left( \frac{2\pi}{\Lambda(\Phi)} \right)^2 = \int_{\Lambda} \left( \frac{2\pi}{\Lambda(\Phi_{\lambda})} \right)^2 dP(\lambda).$$

*Proof.* Let me write  $\Gamma$  for the covariance function of  $\Phi$ ,  $\varphi_{\lambda}$  for the spherical function with spectral parameter  $\lambda$ . Note that  $\Gamma = \int_{\Lambda} \varphi_{\lambda} dP(\lambda)$  as we saw, and taking up the notations of the proof of Lemma 4.1, recall that

$$\left( \frac{2\pi}{\Lambda(\Phi)} \right)^2 = L_{\tilde{\gamma}}^2(\Gamma).$$

I just need to evaluate  $L_{\tilde{\gamma}}^2(\Gamma)$ . But of course switching with the integration with respect to  $\lambda$  yields

$$L_{\tilde{\gamma}}^2(\Gamma)(x_0) = \int_{\Lambda} L_{\tilde{\gamma}}^2(\varphi_{\lambda})(x_0) dP(\lambda),$$

and the lemma follows. □

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<sup>17</sup>Relating this to the geometry of  $X$  is a deep question ! See for instance [9], III.D.

**Remark 4.5.** The hypotheses in Lemma 5.2 are of course unnecessarily stringent given the proof, and one can presumably evaluate the typical spacing of a general field on a riemannian homogeneous space  $X$  by using spectral theory to split it into fields with samples in an eigenspace of  $\Delta_X$ .

## 5 Density of zeroes for invariant smooth fields on homogeneous spaces

### 5.1 Statement of the result

In this section, the homogeneous space  $X$  need not be commutative, but need only be riemannian.

Let us start with a definition. Suppose  $\Phi$  is an invariant Gaussian field on  $X$  with values in a finite-dimensional vector space  $V$ . For each  $u$  in  $V$ , the typical spacing  $\Lambda(\langle u|\Phi \rangle)$  of the projection of  $\Phi$  on the axis  $\mathbb{R}u$  depends on the variance  $\beta_u$  of the real-valued Gaussian variable  $\Lambda(\langle u|\Phi(p) \rangle)$  (here  $p$  is any point of  $X$ ), but  $\sqrt{\beta_u}\Lambda(\langle u|\Phi \rangle)$  does not depend on  $u$ . Choosing an orthonormal basis  $(u_1, \dots, u_{\dim V})$  of  $V$ , we can form the quantity  $\prod_{i=1}^{\dim V} \sqrt{\beta_{u_i}}\Lambda(\langle u_i|\Phi \rangle)$ ; it does not depend on the chosen basis, I will call it the *volume of an elementary cell* for  $\Phi$ , and write  $\mathcal{V}(\Phi)$  for it.

The terminology is transparent if  $\dim V$  and  $\dim X$  coincide, provided  $\Phi(p)$  is an isotropic Gaussian vector and  $\beta_u$  equals 1 for each  $u$ . The notion corresponds to the notion of *hypercolumn* from neuroscience (see [19] for the biological definition, [31] for its geometrical counterpart).

**Theorem 5.1.** *Suppose  $\Phi$  is a smooth, invariant Gaussian random field on  $X$  with values in  $\mathbb{R}^{\dim X}$ . Write  $\mathcal{N}_A$  for the random variable recording the number of zeroes of  $\Phi$  in a Borel region  $A$  of  $X$ , and  $\text{Vol}(A)$  for its volume (measured using the  $G$ -invariant metric introduced above). Write  $\mathcal{V}(\Phi)$  for the volume of an elementary cell for  $\Phi$ . Then*

$$\mathbb{E}(\mathcal{N}_A) \frac{\mathcal{V}(\Phi)}{\text{Vol}(A)} = (\dim X)! \left( \frac{\pi}{2} \right)^{(\dim X)/2}.$$

**Remark 5.2.** My reason for stating Theorem 5.1 on its own, even though it is a special case of Theorem 3 below, is that the two-dimensional result which motivated this study is one in which it is natural to have  $\dim X = 2$  and  $V = \mathbb{C}$ , and that Theorem 5.1 is a neatly stated generalization to higher dimensions.

★

Theorem 5.1 can be extended to a result on the volume of the zero-set of Gaussian fields with values in a Euclidean space of any dimension, as follows. If  $\Phi$  is a smooth invariant Gaussian field on a symmetric space  $X$  with values in a finite-dimensional space  $V$ , then the zero-set of  $\Phi$  is generically a union of  $(\dim X - \dim V)$ -dimensional submanifolds (and is generically empty if  $\dim V > \dim X$ ). Every submanifold of  $X$  inherits a metric, and hence a volume form, from that of  $X$ , and this almost surely gives a meaning to the volume of the intersection of  $\Phi^{-1}(0)$  with a compact subset of  $X$ . When  $A$  is a Borel region of  $X$  and  $u$  is an element of  $V$ , we can thus define a real-valued random variable  $\mathcal{M}_{\Phi,A}(u)$  by

recording the volume of  $A \cap \Phi^{-1}(u)$  for all samples of  $\Phi$  for which  $u$  is a regular value, and recording, say, zero for all samples of  $\Phi$  for which  $u$  is a singular value.

**Theorem 5.3.** *Suppose  $\Phi$  is a reduced invariant Gaussian random field on a homogeneous space  $X$  with values in a Euclidean space  $V$ . Write  $\mathcal{M}_{\Phi,A}$  for the random variable recording the geometric measure of  $\Phi^{-1}(0)$  in a Borel region  $A$  of  $X$ , and  $\text{Vol}(A)$  for the volume of  $A$ . Write  $\mathcal{V}(\Phi)$  for the volume of an elementary cell for  $\Phi$ . Then*

$$\mathbb{E}(\mathcal{M}_{\Phi,A}) \cdot \frac{\mathcal{V}(\Phi)}{\text{Vol}(A)} = \frac{(\dim X)!}{(\dim X - \dim V)!} \cdot \left(\frac{\pi}{2}\right)^{(\dim V)/2}$$

Theorem 2 obviously implies Theorem 1 if we take as a convention that  $\mathcal{M}_{\Phi,A}(u)$  is  $N(A, u)$  when  $\dim X$  and  $\dim V$  coincide.

**Remark 5.4.** Thus, in the unit provided by the volume of an elementary cell, the density of the zero-set in an invariant field depends only on the dimension of the source and target spaces, and *not on the group acting*. Of course the group structure is quite relevant for determining the appropriate unit, as we saw.

**Remark 5.5.** I should remark here that when  $\dim X$  and  $\dim V$  do not coincide, the volume unit  $\mathcal{V}(\Phi)$  is not the volume of anything  $\dim X$ -dimensional in any obvious way – but  $\mathbb{E}(\mathcal{M}_{\Phi,A})$  is not, either. It is Theorem 5.2 that makes it natural to interpret  $\mathcal{V}(\Phi)$  as a volume unit.

## 5.2 Proof of Theorem 5.2

I will use Azais and Wschebor's Kac-Rice formula for random fields (Theorem 6.8 in [5]); the proof of Theorem 5.2 will be a rather direct adaptation of the one which appears for complex-valued fields on the Euclidean plane and space in [6], [7].

Let me recall their formula, adding a trivial adaptation to our situation where the base space is a riemannian manifold rather than a Euclidean space.

**Lemma 5.1.** *Suppose  $(M, g)$  is a riemannian manifold, and  $\Phi : M \rightsquigarrow \mathbb{R}^{\dim M}$  is a smooth Gaussian random field. Assume that the variance of the Gaussian vector  $\Phi(p)$  at each point  $p$  in  $M$  is nonzero.*

*For each  $u$  in  $\mathbb{R}^{\dim M}$  and every Borel subset  $A$  in  $M$ , write  $N(A, u)$  for the random variable recording the number of points in  $\Phi^{-1}(u)$ .*

*Then as soon as  $\mathbb{P}\{\exists p \in M, \Phi(p) = u \text{ and } \det[d\Phi(p)] = 0\} = 0$ ,*

$$\mathbb{E}[N(A, u)] = \int_A \mathbb{E}\left\{|\det[d\Phi(p)d\Phi(p)^\dagger]|^{1/2} \|\Phi(p) = u\} p_{\Phi(p)}(u) d\text{Vol}_g(p) \quad (3)$$

*Proof.* After splitting  $A$  into a suitable number of Borel subsets, I can obviously work in a single chart and assume that  $A$  is contained in an open subset  $U$  of  $M$  for which there is a diffeomorphism  $\psi : M \supset U \rightarrow \psi(U) \subset \mathbb{R}^{\dim M}$ . I turn  $\Phi|_U$  into a Gaussian random field  $\Psi$  on  $\mathbb{R}^{\dim M}$  by setting

$$\Psi \circ \psi = \Phi.$$

Then I can apply Theorem 6.2 in [5] to count the zeroes of  $\Psi$  in  $\psi(A)$ ; since there are as many zeroes of  $\Psi$  in  $\psi(A)$  as there are zeroes of  $\Phi$  in  $A$ , the theorem yields

$$\mathbb{E}[N(A, u)] = \int_{\psi(A)} \mathbb{E}\left\{|\det[d\Psi(x)d\Psi(x)^\dagger]|^{1/2} \|\Psi(x) = u\} p_{\Psi(x)}(u) dx,$$

where the volume element is Lebesgue measure.

Now, let us start from the right-hand-side of (3) and change variables using  $\psi$ ; we get

$$\begin{aligned} \int_A \mathbb{E} \{ |\det [d\Phi(p)d\Phi(p)^\dagger]| \mid \Phi(p) = u \} p_{\Phi(p)}(u) dVol_g(p) &= {}^{18} \\ \int_{\psi(A)} \mathbb{E} \{ |\det [d\Phi(\psi^{-1}(x))d\Phi(\psi^{-1}(x))^\dagger]|^{1/2} \mid \Phi(\psi^{-1}(x)) = u \} p_{\Phi(\psi^{-1}(x))}(u) |\det [d\psi^{-1}(x)]| dx &= {}^{19} \\ \int_{\psi(A)} \mathbb{E} \{ |\det [d\psi^{-1}(x)] \det [d\Phi(\psi^{-1}x)] \det [d\Phi(\psi^{-1}x)^\dagger]| \det [d\psi^{-1}(x)^\dagger]|^{1/2} \mid \Psi(x) = u \} p_{\Psi(x)}(u) dx &= {}^{20} \\ \int_{\psi(A)} \mathbb{E} \{ |\det [d\Psi(x)d\Psi(x)^\dagger]|^{1/2} \mid \Psi(x) = u \} p_{\Psi(x)}(u) dx &= N(A, u) \end{aligned}$$

as announced.  $\square$

Let us return to the case where  $\Phi$  is an invariant Gaussian field on a homogeneous space. Choose an orthonormal basis  $(u_1, \dots, u_{\dim V})$  of  $V$ , write  $\beta_i$  for the standard deviation of the Gaussian variable  $\langle u_i, \Phi(p) \rangle$  at each  $p$  (which does not depend on  $p$ , and  $\mathcal{V}$  for the quantity  $\beta_1 \dots \beta_{\dim V}$ , which is the volume of the characteristic ellipsoid for the Gaussian vector  $\Phi(p)$  at each  $p$  and depends neither on  $p$  nor on the choice of basis in  $V$ ).

To prove Theorem 2 we need to look for  $N(A, 0)$ , and since the field  $\Phi$  is Gaussian, we know that  $p_{\Phi(p)}(0) = \mathcal{V}(2\pi)^{-(\dim V)/2}$  for each  $p$ . In addition, because of the invariance we know that  $p \mapsto \mathbb{E}[\Phi(p)^2]$  is a constant function on  $X$ , so for any vector field  $\tilde{\gamma}$  on  $X$ ,

$$\mathbb{E}[(L_{\tilde{\gamma}}\Phi)(p)\Phi(p)] = 0.$$

A first consequence is that  $\mathbb{P}\{\exists p \in M, \Phi(p) = 0 \text{ and } \det[d\Phi(p)d\Phi(p)^\dagger] = 0\}$  is indeed zero, and that we can use Lemma 5.1. Another consequence is that if we choose a basis in  $T_p X$  and view  $d\Phi(p)$  as a matrix, the entries will be Gaussian random variables which are independent from every component of  $\Phi(p)$ . This means we can remove the conditioning in (3). Thus,

$$\mathbb{E}[N(A, 0)] = \frac{1}{(2\pi\mathcal{V}^2)^{(\dim X)/2}} \int_A \mathbb{E}\{|\det[d\Phi(p)d\Phi(p)^\dagger]|\} dVol_g(p). \quad (4)$$

Now,  $d\Phi(p)$  is a random endomorphism from  $T_p X$  to  $V$ . Recall that if  $\gamma$  is a tangent vector to  $X$  at  $p$ , the probability distribution of  $(L_\gamma\Phi)(p)$ , a Gaussian random vector in  $V$ , does not depend on  $p$ , and does not depend on  $\gamma$ . Thus there is a basis  $(v_1, \dots, v_{\dim V})$  of  $V$  such that for each  $\gamma$  in  $T_p X$ ,  $\langle (L_\gamma\Phi)(p), v_i \rangle$  is independent from  $\langle (L_\gamma\Phi)(p), v_j \rangle$  if  $i \neq j$  (the  $v_i$ s generate the principal axes for  $(L_\gamma\Phi)(p)$ ). If we choose any basis of  $T_p X$  and write down the corresponding matrix for  $d\Phi(p)$  (it has  $\dim X$  rows and  $\dim V$  columns), then the columns will be independent and will be isotropic Gaussian vectors in  $\mathbb{R}^{\dim X}$ .

To go further, we need the following simple remark.

**Lemma 5.2.** *Suppose  $M$  is a matrix with  $n$  rows and  $k$  columns,  $n \geq k$ , and write  $(m_1, \dots, m_k) \in (\mathbb{R}^n)^k$  for its columns. Then the determinant of  $MM^\dagger$  is the square of the volume of the parallelotope  $\left\{ \sum_{i=1}^k t_i m_i \mid t_i \in [0, 1] \right\}$ .*

*Proof.* Choose an orthonormal basis  $(m_{k+1}, \dots, m_n)$  of  $\text{Span}(m_1, \dots, m_k)^\perp$ . Then the signed volume of the  $k$ -dimensional parallelotope  $\left\{ \sum_{i=1}^k t_i m_i \mid t_i \in [0, 1] \right\}$  is the same as that of the  $n$ -dimensional parallelotope  $\left\{ \sum_{i=1}^n t_i m_i \mid t_i \in [0, 1] \right\}$ .

Write  $\tilde{M}$  for the  $n \times n$  matrix whose columns are the coordinates of the  $m_i$  in the canonical basis of  $\mathbb{R}^n$ . Then  $\tilde{M}\tilde{M}^\dagger$  is block-diagonal, one block is  $MM^\dagger$  and the other block is the identity because  $(m_{k+1}, \dots, m_n)$  is an orthonormal family.

Thus the determinant of  $MM^\dagger$  is the square of that of  $\tilde{M}$ , and  $\det(\tilde{M})$  is the volume of the parallelotope  $\left\{ \sum_{i=1}^n \alpha_i m_i \mid \alpha_i \in [0, 1] \right\}$ .  $\square$

Coming back to the proof of Theorem 2, we are left with evaluating the mean Hausdorff volume of the random parallelotope generated by  $\dim V$  independent isotropic Gaussian vectors in  $\mathbb{R}^{\dim X}$ .

**Lemma 5.3.** *Suppose  $u_1, \dots, u_k$  are independent isotropic Gaussian vectors with values in  $\mathbb{R}^n$ , so that the probability distribution of  $u_i$  is  $x \mapsto \frac{1}{\alpha_i \sqrt{2\pi}} e^{-\|x\|^2/2\alpha_i^2}$ . Write  $\mathcal{V}$  for the characteristic volume  $\alpha_1 \dots \alpha_k$ , and write  $\mathbf{V}$  for the random variable recording the  $k$ -dimensional volume of the parallelotope  $\left\{ \sum_{i=1}^k t_i u_i \mid t_i \in [0, 1] \right\}$ . Then*

$$\mathbb{E}[\mathbf{V}] = \frac{n!}{(n-k)!} \mathcal{V}.$$

*Proof.* Let me start with  $k$  (deterministic) vectors in  $\mathbb{R}^n$ , say  $u_1^0, \dots, u_k^0$ , and choose a basis  $u_{k+1}^0, \dots, u_n^0$  for  $\text{Span}(u_1^0, \dots, u_n^0)^\perp$ . Since  $\det(u_1^0, \dots, u_k^0) = \det(u_1^0, \dots, u_n^0)$  is the (signed) volume of the parallelotope generated by the  $u_i^0$ s, we can use the "base times height" formula: writing  $P_V$  for the orthogonal projection from  $\mathbb{R}^n$  onto a subspace  $V$ ,

$$\text{Vol}(u_1^0, \dots, u_n^0) = \|P_{\text{Span}(u_2^0, \dots, u_n^0)^\perp}(u_1^0)\| \text{Vol}(u_2^0, \dots, u_n^0).$$

Of course then

$$\text{Vol}(u_1^0, \dots, u_n^0) = \prod_{i=1}^k \|P_{\text{Span}(u_{i+1}^0, \dots, u_n^0)^\perp}(u_i^0)\|.$$

Let me now return to the situation with random vectors. Because  $u_1, \dots, u_k$  are independent, the above formula becomes

$$\mathbb{E}[\text{Vol}(u_1, \dots, u_k)] = \prod_{i=1}^k \mathbb{E}[N(u_i, V^i)]$$

where  $N(u_i, V^i)$  is the random variable recording the norm of the projection of  $u_i$  on any  $(i)$ -dimensional subspace of  $\mathbb{R}^n$ . The projection is a Gaussian vector, and so its norm has a chi-squared distribution with  $i$  degrees of freedom. Given the probability distribution of  $u_i$ , the expectation for the norm is then  $i\alpha_i$ , and this does prove Lemma 5.3.  $\square$

To complete the proof of Theorem 5.2, choose an orthonormal basis  $(\gamma_1, \dots, \gamma_n)$  in  $T_p X$ . Apply Lemma 5.3 to the family  $((L_{\gamma_i} \langle v_i, \Phi \rangle)(p))_{i=1..n}$ . Then (4) becomes

$$\mathbb{E}[N(A, 0)] = \frac{1}{(2\pi)^{(\dim V)/2} \mathcal{V}} \text{Vol}(A) \frac{(\dim X)!}{(\dim X - \dim V)!} \prod_{i=1}^{\dim V} \mathbb{E}[(L_{\gamma_1}(\langle v_i, \Phi \rangle)(x_0))^2]^{1/2}.$$

To bring the typical spacing back into the picture, recall that the definition and the Kac-Rice formula (2) say that  $\mathbb{E}[(L_{\gamma_1}(\langle v_i, \Phi \rangle)(x_0))^2]^{1/2}$  is none other than  $\frac{\pi}{\Lambda(\langle v_i, \Phi \rangle)}$ . Thus

$$\mathbb{E}[N(A, 0)] \frac{\mathcal{V} \prod_{i=1}^d \Lambda(\langle v_i, \Phi \rangle)}{\text{Vol}(A)} = \frac{\pi^{\dim V}}{(2\pi)^{(\dim V)/2}} \frac{(\dim X)!}{(\dim X - \dim V)!}.$$

and since  $\mathcal{V} \prod_{i=1}^d \Lambda(\langle v_i, \Phi \rangle)$  is the volume of an elementary cell for  $\Phi$ , Theorem 2 is established. □

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